

CIRCULAR LAW FOR RANDOM MATRICES WITH UNCONDITIONAL LOG-CONCAVE DISTRIBUTION

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ABSTRACT. We explore the validity of the circular law for random matrices with non i.i.d. entries. Let A be a random $n \times n$ real matrix having as a random vector in $\mathbb{R}^{n \times n}$ a log-concave isotropic unconditional law. In particular, the entries are uncorellated and have a symmetric law of zero mean and unit variance. This allows for some dependence and non equidistribution among the entries, while keeping the special case of i.i.d. standard Gaussian entries. Our main result states that as n goes to infinity, the empirical spectral distribution of $\frac{1}{\sqrt{n}}A$ tends to the uniform law on the unit disc of the complex plane.

1. INTRODUCTION

For an $n \times n$ matrix A , denote by ν_A its spectral measure, defined as $\nu_A = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A which are the roots in \mathbb{C} of the characteristic polynomial of A , counted with multiplicities, and where δ_x stands for the Dirac mass at point x . If A is Hermitian, then we will treat ν_A as a finite discrete probability measure on \mathbb{R} , while in the general case it is a finite discrete probability measure on \mathbb{C} . When A is a random matrix, then ν_A becomes a random measure.

The behaviour of the spectral measure of non-Hermitian random matrices has drawn considerable attention over the years following the work by Mehta [32], who proved, by using explicit formulas due to Ginibre [18], that the expected spectral measure of $n \times n$ matrices with i.i.d. standard complex Gaussian entries scaled by \sqrt{n} converges to the uniform measure on the unit disc (which we will call the circular law). Further developments [19, 20, 7, 36, 21, 41] succeeded in extending the result to a beautiful universal statement valid for any random matrix with i.i.d. entries of unit variance.

It is natural to try to relax the conditions of finite variance and/or independence imposed on the entries of the matrix by those results. Relaxing the finite variance assumption leads to limiting distributions which are not the circular law, see [10, 12]. There are various ways to relax the independence assumption. The circular law was first proved for various models of random matrices with i.i.d. rows, such as for certain random Markov matrices [11], for random matrices with i.i.d. log-concave rows [1, 2], for random ± 1 matrices with i.i.d. rows of given sum [35] (see also [40]), etc. Beyond the i.i.d. rows structure, the circular law was proved only for very specific models such as for blocs of Haar unitary matrices [25, 16], and more recently for random doubly stochastic matrices following the uniform distribution on the Birkhoff polytope [34]. Our main result stated in Theorem 1.1 below is establishing the circular law for a new class of random matrices with dependent entries beyond the i.i.d. rows structure. Theorem 1.1 is a natural extension of the model studied in [1]. However, it does not include as special cases the models studied in [25, 16, 34]. The general idea behind Theorem 1.1 comes from asymptotic geometric analysis and states

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roughly that in large dimensions, for many aspects, unconditional isotropic log-concave measures behave like product measures with unit variance and sub-exponential tail.

We say that a probability measure μ on \mathbb{R}^d is *log-concave* when for all nonempty compact sets A, B and $\theta \in (0, 1)$, $\mu(\theta A + (1 - \theta)B) \geq \mu(A)^\theta \mu(B)^{1-\theta}$. A measure μ not supported on a proper affine subspace of \mathbb{R}^d is log-concave iff it has density e^{-V} where $V: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is a convex function, see [13]. We say that μ is *isotropic* if it is centered and its covariance matrix is the identity, in other words if the coordinates are uncorellated, centered, and have unit variance. Recall finally that μ is called *unconditional* if $X = (X_1, \dots, X_d)$ and $(\varepsilon_1 X_1, \dots, \varepsilon_d X_d)$ have the same law when X has law μ and $\varepsilon_1, \dots, \varepsilon_d$ are i.i.d. Rademacher random variables (signs) independent of X . The isotropy and the unconditionality are related to the canonical basis of \mathbb{R}^d , while the log-concavity is not. Note that unconditionality together with unit variances imply automatically isotropy. We use in the sequel the identifications $\mathcal{M}_n(\mathbb{R}) \equiv \mathbb{R}^{n^2}$ and $\mathcal{M}_n(\mathbb{C}) \equiv \mathbb{C}^{n^2}$ in which a $n \times n$ matrix M with rows R_1, \dots, R_n is identified with the vector (R_1, \dots, R_n) .

Theorem 1.1 (Circular law for isotropic unconditional log-concave random matrices). *Let $A_n = [X_{ij}^{(n)}]_{1 \leq i, j \leq n}$ be $n \times n$ random matrices, defined on a common probability space. Assume that for each n , the distribution of A_n as a random vector in \mathbb{R}^{n^2} is log-concave, isotropic and unconditional. Then with probability one the empirical spectral measure of $\frac{1}{\sqrt{n}}A_n$ converges weakly as $n \rightarrow \infty$ to the uniform measure on the unit disc of \mathbb{C} .*

If one drops the unconditionality assumption in Theorem 1.1 then the limiting spectral measure needs not be the circular law. For instance, one may consider random matrices with density of the form $A \mapsto \exp(-\text{Tr}(V(\sqrt{AA^*})))$ with V convex and increasing, which are log-concave (Klein's lemma) but for which the limiting spectral distribution depends on V , see [17, eq. (5.8) p. 654] and [23]. This is in contrast with the model of random matrices with i.i.d. log-concave rows studied in [1], for which it turned out that the circular law holds without assuming unconditionality, as explained in [2].

We prove Theorem 1.1 by using the by now classical Hermitization method introduced by Girko in [19] and further developed by Tao and Vu in [41]. Following the scheme presented in [12], we first establish the convergence of the spectral measure of the matrix

$$B_n(z) := \sqrt{\left(\frac{1}{\sqrt{n}}A_n - z\text{Id}\right)\left(\frac{1}{\sqrt{n}}A_n - z\text{Id}\right)^*} \quad (1)$$

and later obtain bounds on the small singular values of the matrix, which will allow us to prove almost sure uniform integrability of the logarithm with respect to the empirical spectral measure of $B_n(z)$. Putting these ingredients together we obtain convergence of the logarithmic potential of the empirical spectral measure of $\frac{1}{\sqrt{n}}A_n$, which ends the proof.

Outline. The organization of the paper is as follows. In Section 2, we first recall some basic results concerning log-concave unconditional measures, which will be useful in the proof. Next in Section 3 we give the outline of the argument, prove convergence of the empirical spectral measure of $B_n(z)$ and reduce the proof of Theorem 1.1 to lower bounds on the singular values of A_n . These are proved in Section 4.

Notations. We will denote by C, c positive absolute constants and by C_a, c_a constants depending only on the parameter a . In both cases the values of constants may differ between occurrences (even in the same line). By $|\cdot|$ we will denote the standard Euclidean norm of a vector in \mathbb{C}^n or \mathbb{R}^n . We will use $\|\cdot\|$ to denote the operator norm of a matrix and $\|\cdot\|_{\text{HS}}$ to denote its Hilbert-Schmidt norm. For a probability measure μ on \mathbb{R} and ξ

in $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ by $m_\mu(\xi)$ we will denote the Cauchy-Stieltjes transform, i.e.

$$m_\mu(\xi) = \int_{\mathbb{R}} \frac{1}{\lambda - \xi} \mu(d\lambda).$$

We refer to [4, 6] for general theory of Cauchy-Stieltjes transforms and in particular their connection with weak convergence. For an $n \times n$ matrix A by $s_1(A) \geq \dots \geq s_n(A)$ we will denote its singular values, i.e. eigenvalues of $(AA^*)^{1/2}$.

2. BASIC FACTS ON UNCONDITIONAL LOG-CONCAVE MEASURES

The geometric and probabilistic analysis of log-concave measures is a well developed area of research. We recommend the reader the forthcoming monograph [14] for a detailed presentation of this rich theory. What we will need for our analysis is properties related to the behaviour of densities and concentration of measure results. In general such questions are difficult and related to famous open problems, like the Kannan-Lovasz-Simonovits question [26] or the slicing problem [24, 33].

Fortunately, unconditional log-concave measures behave in a much more rigid way than general ones, in particular some of the aforementioned questions have been answered either completely or up to terms which are logarithmic in the dimension and as such do not cause difficulties in the problems we are about to deal with. Below we present the ingredients we will use throughout the proof.

The first fact we will need follows immediately from the definition of log-concavity: linear images of log-concave random vectors are themselves log-concave. We also have the following tail estimate, which is a special case of a more general fact due to Borell [13].

Theorem 2.1 (Log-concave random variables have sub-exponential tails). *If X is a log-concave, mean zero, variance one random variable, then for all $t \geq 0$,*

$$\mathbb{P}(|X| \geq t) \leq 2 \exp(-ct).$$

The next theorem provides a positive answer to the so-called slicing conjecture in the case of unconditional log-concave measures. Let us recall that the conjecture (in one of many equivalent formulations) asserts that the density of a log-concave isotropic measure in \mathbb{R}^n is bounded by C^n . While this question is wide open in the general case, it has been answered positively in the case of unconditional measures. We refer for instance to [9] for a proof. We will use it to obtain sharp small ball inequalities, which will be useful when dealing with dependence between different rows of the matrix.

Theorem 2.2 (Density bound for log-concave measures). *The density of a log-concave unconditional measure in \mathbb{R}^n is bounded from above by C^n , where C is a universal constant.*

Another result we will need is the following version of the Poincaré inequality for log-concave measures, together with the concentration of measure inequality which follows from it. We refer to [27, 8] and to the books [30, 5] for the general theory of concentration of measure and its relations with functional inequalities. The question whether all isotropic log-concave measures satisfy the Poincaré inequality with a universal constant is another famous open problem in asymptotic geometric analysis [26].

Theorem 2.3 (Poincaré inequality from unconditional log-concavity). *If X is an isotropic unconditional log-concave random vector in \mathbb{R}^n , then for every smooth $f: \mathbb{R}^n \rightarrow \mathbb{R}$,*

$$\text{Var } f(X) \leq C \log^2(n+1) \mathbb{E} |\nabla f(X)|^2.$$

The above theorem implies in particular concentration of measure inequality for Lipschitz functions via the so called Herbst argument, see for instance [22] and [30].

Theorem 2.4 (Concentration of measure from the Poincaré inequality). *If a random vector X in \mathbb{R}^n satisfies the Poincaré inequality $\text{Var } f(X) \leq \lambda^{-1} \mathbb{E} |\nabla f(X)|^2$ for all smooth functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$, then for all 1-Lipschitz functions $g: \mathbb{R}^n \rightarrow \mathbb{R}$ and all $t > 0$,*

$$\mathbb{P}(|g(X) - \mathbb{E}g(X)| \geq t) \leq 2 \exp(-c\sqrt{\lambda}t).$$

Finally we will need the following result taken from [29], built on previous developments in [9]. It provides a comparison of norms of log-concave unconditional random vectors with norms of vectors with independent exponential coordinates.

Theorem 2.5 (Comparison of tails for unconditional log-concave measures). *If X is an isotropic unconditional log-concave random vector in \mathbb{R}^n and \mathcal{E} a standard n -dimensional symmetric exponential vector (i.e. with i.i.d. components of symmetric exponential distribution with variance one), then for any seminorm $\|\cdot\|$ on \mathbb{R}^n and any $t > 0$,*

$$\mathbb{P}(\|X\| \geq Ct) \leq C\mathbb{P}(\|\mathcal{E}\| \geq t).$$

3. PROOF OF THE MAIN RESULT BY REDUCTION TO SINGULAR VALUES BOUNDS

As already mentioned in the introduction, we will follow the Hermitization method introduced by Girko, together with a Tao and Vu approach to obtain lower bounds on the singular values. We refer the reader to [12] for a presentation of this method in the general case. Let $B_n(z)$ be as in (1). Thanks to [12, Lemma 4.3], to prove that with probability one $\nu_{n^{-1/2}A_n}$ converges weakly to the uniform measure on the unit disc, it is enough to demonstrate the following two assertions.

- (i) For all $z \in \mathbb{C}$ the spectral measure $\nu_{z,n} := \nu_{B_n(z)}$ converges almost surely to some deterministic probability measure ν_z on \mathbb{R}_+ . Moreover, for almost all $z \in \mathbb{C}$,

$$U(z) := - \int_{\mathbb{R}_+} \log(s) \nu_z(ds) = \begin{cases} -\log|z| & \text{if } |z| > 1 \\ \frac{1}{2}(1 - |z|^2) & \text{otherwise;} \end{cases}$$

- (ii) For all $z \in \mathbb{C}$, with probability one the function $s \mapsto \log(s)$ is uniformly integrable with respect to the family of measures $\{\nu_{z,n}\}_{n \geq 1}$.

The quantity $U_n(z) := \int_{\mathbb{C}} \log \frac{1}{|\lambda - z|} \nu_{\frac{1}{\sqrt{n}}A_n}(d\lambda)$ is the logarithmic potential of $\nu_{\frac{1}{\sqrt{n}}A_n}$, and we have $U_n(z) = \int_0^\infty \log(s) \nu_{B_n(z)}(ds)$, see [12]. We will first prove point (i).

Proposition 3.1 (Singular values of shifts). *Assertion (i) above is true.*

Proof. Let us fix $z \in \mathbb{C}$. From Theorem 2.3 and Theorem 2.4 it follows easily that the Euclidean length of a random row/column of A_n , normalized by \sqrt{n} , converges in probability to one. Thus we deduce by [1, Theorem 2.4] that the expected spectral measure $\mathbb{E}\nu_{z,n}$ converges weakly, as $n \rightarrow \infty$, to a probability measure ν_z which depends only on z (the identification of ν_z and the formula involving U can be then done and checked on the case of i.i.d. Gaussian entries, see for instance [12]). The rest of our proof is now devoted to the upgrade to almost sure convergence, by using concentration of measure for the Cauchy-Stieltjes transform. For now, we know that for every $\xi \in \mathbb{C}_+$,

$$\mathbb{E}m_{\nu_{z,n}}(\xi) = \mathbb{E} \int_{\mathbb{R}_+} \frac{1}{\lambda - \xi} \nu_{z,n}(d\lambda) = m_{\mathbb{E}\nu_{z,n}}(\xi) \xrightarrow{n \rightarrow \infty} m_{\nu_z}(\xi).$$

At this step, we observe that if C and C' are in $\mathcal{M}_n(\mathbb{C})$ with singular values $s_1 \geq \dots \geq s_n$ and $s'_1 \geq \dots \geq s'_n$ respectively then for every $\xi \in \mathbb{C}_+$,

$$\begin{aligned} |m_{\nu_{\sqrt{CC^*}}}(\xi) - m_{\nu_{\sqrt{C'C'^*}}}(\xi)| &= \left| \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i - \xi} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s'_i - \xi} \right| \\ &\leq \frac{1}{n} \sum_{i=1}^n \left| \frac{s_i - s'_i}{(s_i - \xi)(s'_i - \xi)} \right| \\ &\leq \frac{1}{n|\Im \xi|^2} \sum_{i=1}^n |s_i - s'_i| \\ &\leq \frac{1}{\sqrt{n}|\Im \xi|^2} \sqrt{\sum_{i=1}^n |s_i - s'_i|^2} \\ &\leq \frac{1}{\sqrt{n}|\Im \xi|^2} \|C - C'\|_{\text{HS}}, \end{aligned}$$

where the last step follows from the Hoffman-Wielandt inequality for singular values, see for instance [15, Chapter 4]. Thus both the real and the imaginary parts of $m_{\nu_{z,n}}$ are $1/(n(\Im \xi)^2)$ Lipschitz functions of A_n , with respect to the Hilbert-Schmidt norm, which is the Euclidean norm on $\mathcal{M}_n(\mathbb{R}) \equiv \mathbb{R}^{n^2}$. Therefore, by Theorem 2.3 and Theorem 2.4, we get, for every $\xi \in \mathbb{C}_+$ and $\varepsilon > 0$,

$$\mathbb{P}(|m_{\nu_{z,n}}(\xi) - \mathbb{E}m_{\nu_{z,n}}(\xi)| \geq \varepsilon) \leq 2 \exp(-cn\varepsilon \Im(z)^2 / \log(n)).$$

Now by the first Borel-Cantelli lemma, with probability one, $m_{\nu_{z,n}}(\xi) - \mathbb{E}m_{\nu_{z,n}}(\xi) \rightarrow 0$ as $n \rightarrow \infty$ (the set of probability one depends on ξ). Since $\mathbb{E}m_{\nu_{z,n}}(\xi) \rightarrow m_{\nu_z}(\xi)$ as $n \rightarrow \infty$, we get that with probability one, $m_{\nu_{z,n}}(\xi) \rightarrow m_{\nu_z}(\xi)$ as $n \rightarrow \infty$. Since the Cauchy-Stieltjes transform is uniformly continuous on every compact subset of \mathbb{C}_+ , it follows that with probability one, for every $\xi \in \mathbb{C}_+$, $m_{\nu_{z,n}}(\xi) \rightarrow m_{\nu_z}(\xi)$ as $n \rightarrow \infty$, which implies finally that with probability one, $\nu_{z,n}$ converges weakly to ν_z as $n \rightarrow \infty$. \square

To finish the proof of Theorem 1.1 it is thus enough to demonstrate (ii). We will do this using the following three lemmas which give bounds on singular values of the matrix $\frac{1}{\sqrt{n}}A_n - z\text{Id}$. The proofs of the lemmas will be deferred to the next section. Let us remark that the formulations we present are in fact more general than what is needed for the proof of Theorem 1.1. We also recall that by C, c we denote absolute constants.

The first lemma estimates the operator norm of the matrix (largest singular value).

Lemma 3.2 (Largest singular value). *Let A_n be an $n \times n$ random matrix with log-concave isotropic unconditional distribution and let M_n be a deterministic $n \times n$ matrix with $\|M_n\| \leq R\sqrt{n}$ for some $R > 0$. Then for all $t \geq 1$,*

$$\mathbb{P}(\|A_n + M_n\| \geq (R + C)\sqrt{n} + t) \leq 2 \exp(-ct).$$

Our next lemma provides a bound on the smallest singular value.

Lemma 3.3 (Smallest singular value). *Let A_n be an $n \times n$ random matrix with log-concave isotropic unconditional distribution and let M_n be a deterministic $n \times n$ matrix. Then*

$$\mathbb{P}(s_n(A_n + M_n) \leq n^{-6.5}) \leq Cn^{-3/2}.$$

Remark 3.4. *The above lemma is certainly suboptimal, but it is sufficient for our applications. In view of the results for Gaussian matrices [38] it is natural to conjecture that for $\varepsilon \in (0, 1)$, $\mathbb{P}(s_n(A_n + M_n) \leq \varepsilon n^{-1/2}) \leq C\varepsilon$. For a matrix A_n with independent log-concave isotropic columns it is proven in [3] that $\mathbb{P}(s_n(A_n) \leq \varepsilon n^{-1/2}) \leq C\varepsilon \log^2(2/\varepsilon)$.*

The next lemma we will need gives a bound on the singular values s_{n-i} , where $i > n^\gamma$ for some $\gamma \in (0, 1)$. It is analogous to an estimate in [41] (see also [12]) used to prove the circular law in the i.i.d. case under minimal assumptions. The main difficulty in its proof in our setting is lack of independence between the rows of A_n , which has to be replaced by unconditionality and geometric properties implied by log-concavity.

Lemma 3.5 (Count of small singular values). *Let A_n be an $n \times n$ random matrix with log-concave isotropic unconditional distribution and M_n a deterministic $n \times n$ matrix with $\|M_n\| \leq R\sqrt{n}$. Let also $\gamma \in (0, 1)$. Then for every $n^\gamma \leq i \leq n-1$,*

$$\mathbb{P}\left(s_{n-i}(n^{-1/2}(A_n + M_n)) \leq c_R \frac{i}{n}\right) \leq 2 \exp(-c_{R,\gamma} n^{\gamma/3}).$$

Let us now finish the proof of Theorem 1.1, by demonstrating how the above lemmas imply point (ii). By Markov's inequality it is enough to show that for every $z \in \mathbb{C}$, for some small $\alpha > 0$ with probability one

$$\overline{\lim}_{n \rightarrow \infty} \int_0^\infty (s^\alpha + s^{-\alpha}) \nu_{z,n}(ds) < \infty,$$

or in other words, denoting $s_i := s_i(\frac{1}{\sqrt{n}}A_n - z\text{Id})$,

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (s_i^\alpha + s_i^{-\alpha}) < \infty.$$

For all $\alpha \leq 2$ we have, for n large enough,

$$n^{-1} \sum_{i=1}^n s_i^\alpha \leq (n^{-1} \sum_{i=1}^n s_i^2)^{\alpha/2} = (n^{-1/2} \|n^{-1/2} A_n - z\text{Id}\|_{\text{HS}})^\alpha \leq (\|n^{-1/2} A_n - z\text{Id}\|)^\alpha.$$

But by Lemma 3.2 and the Borel-Cantelli lemma, with probability one, we have the bound $\overline{\lim}_{n \rightarrow \infty} \|n^{-1/2} A_n - z\text{Id}\| < C_z$ for some finite constant C_z , and thus

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n s_i^\alpha < \infty.$$

Passing to the other sum, for α, γ small enough, by Lemmas 3.3, 3.5 and the Borel-Cantelli lemma we have with probability one, for some finite constants C_z and $C_{z,\alpha}$,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n s_i^{-\alpha} &= \frac{1}{n} \sum_{i=0}^{\lfloor n^\gamma \rfloor} s_{n-i}^{-\alpha} + \frac{1}{n} \sum_{i=\lfloor n^\gamma \rfloor+1}^{n-1} s_{n-i}^{-\alpha} \\ &\leq \frac{1}{n} n^{7\alpha} n^\gamma + \frac{1}{n} C_z \sum_{i=\lfloor n^\gamma \rfloor+1}^{n-1} \left(\frac{n}{i}\right)^\alpha \\ &\leq n^{7\alpha+\gamma-1} + C_{z,\alpha} n^{\alpha-1} n^{1-\alpha} = \mathcal{O}(1). \end{aligned}$$

This implies (ii) and ends the proof of Theorem 1.1.

4. PROOF OF SINGULAR VALUES BOUNDS

We will start with the proof of Lemma 3.2.

Proof of Lemma 3.2. By the triangle inequality it is enough to estimate $\|A_n\|$. By Theorem 2.5 we can assume that A_n has i.i.d. entries with the standard symmetric exponential distribution. It is well known (see e.g. [28]) that in this case $\mathbb{E}\|A_n\| \leq C\sqrt{n}$. Moreover, by the Poincaré inequality for the product of symmetric exponential measures (see e.g.

[5]) and the fact that the operator norm is 1-Lipschitz with respect to the Hilbert-Schmidt norm, we have $\mathbb{P}(\|A_n\| \geq C\mathbb{E}\|A_n\| + t) \leq \exp(-ct)$, which allows to finish the proof. \square

Before we proceed let us introduce some additional notation to be used in the proofs below. We will assume that $A_n = [X_{ij}^{(n)}]_{i,j \leq n}$ but for notational simplicity we will suppress the superscript (n) and write X_{ij} instead of $X_{ij}^{(n)}$. We will refer to the rows of A_n as X_1, \dots, X_n . Since the law of A_n is unconditional, sometimes we will work with the matrix $[\varepsilon_{ij} X_{ij}]_{i,j \leq n}$, where ε_{ij} are independent Rademacher variables independent of X_{ij} . Slightly abusing the notation, we will sometimes identify this new matrix with A_n .

Let us now pass to the proof of Lemma 3.3.

Proof of Lemma 3.3. We will denote the rows of M_n and $A_n + M_n$ by Y_1, \dots, Y_n and Z_1, \dots, Z_n respectively. By estimates from [37] (see also [10, Lemma B.2]) we have

$$\mathbb{P}(s_n(A_n + M_n) \leq n^{-6.5}) \leq n \max_i \mathbb{P}(\text{dist}(X_i + Y_i, \text{span}(\{Z_j\}_{j \neq i})) \leq n^{-6}). \quad (2)$$

Let us fix i . Remarkably, the conditional distribution of X_i given $(X_j)_{j \neq i}$ is log-concave and unconditional. Let $\sigma_k^2 = \mathbb{E}(X_{ik}^2 | (X_j)_{j \neq i})$. Now for every $\varepsilon > 0$, every random variable X and random vector Y , by Markov's inequality, $\mathbf{1}_{\{\mathbb{E}(X^2 | Y=y) \leq \varepsilon^2\}} \leq \frac{4}{3} \mathbb{P}(|X| \leq 2\varepsilon | Y=y)$ for every y , which gives $\mathbb{P}(\mathbb{E}(X^2 | Y) \leq \varepsilon^2) \leq \frac{4}{3} \mathbb{P}(|X| \leq 2\varepsilon)$, and in particular

$$\mathbb{P}(\sigma_k^2 \leq \varepsilon^2) \leq \frac{4}{3} \mathbb{P}(|X_{ik}| \leq 2\varepsilon).$$

Next, since X_{ik} is log-concave of unit variance, Theorem 2.2 in dimension one gives

$$\mathbb{P}(\sigma_k^2 \leq \varepsilon^2) \leq \frac{4}{3} \mathbb{P}(|X_{ik}| \leq 2\varepsilon) \leq C\varepsilon.$$

Therefore we get

$$\mathbb{P}(\exists_{k \leq n} \sigma_k^2 \leq n^{-7}) \leq Cn \cdot n^{-7/2} = Cn^{-5/2}.$$

Note that $\text{dist}(X_i + Y_i, \text{span}(\{Z_j\}_{j \neq i})) = |\langle X_i + Y_i, e \rangle|$, where e is a random normal to $\text{span}(\{Z_j\}_{j \neq i})$ (note that due to the existence of a density this space is with probability one of dimension $n-1$). Let $e' = \Re e$, $e'' = \Im e$. Since $|e| = 1$, at least one of the real vectors e', e'' has Euclidean length greater than $2^{-1/2}$. Without loss of generality we can assume that $|e'| \geq 2^{-1/2}$ (otherwise we may multiply e by $\sqrt{-1}$). By unconditionality and the fact that e' is measurable with respect to $(X_j)_{j \neq i}$, we get

$$\mathbb{E}(\langle X_i, e' \rangle^2 | (X_j)_{j \neq i}) \geq \frac{1}{2} \min_{k \leq n} \sigma_k^2.$$

Moreover, the conditional distribution of $\langle X_i, e' \rangle$ given $(X_j)_{j \neq i}$ is log-concave and symmetric. Thus we have

$$\begin{aligned} \mathbb{P}(\text{dist}(X_i + Y_i, \text{span}(\{Z_j\}_{j \neq i})) \leq n^{-6}) &= \mathbb{P}(|\langle X_i + Y_i, e \rangle| \leq n^{-6}) \\ &\leq \mathbb{P}(|\langle X_i, e' \rangle - \Re \langle Y_i, e \rangle| \leq n^{-6}) \\ &\leq \mathbb{P}(\exists_{k \leq n} \sigma_k^2 \leq n^{-7}) + Cn^{-6} \mathbb{E} \left(\mathbb{E}(\langle X_i, e' \rangle^2 | (X_j)_{j \neq i}) \right)^{-1/2} \mathbf{1}_{\{\forall_k \sigma_k^2 > n^{-7}\}} \\ &\leq Cn^{-5/2} + Cn^{-6} n^{7/2} \leq Cn^{-5/2}, \end{aligned}$$

where we used conditionally the fact that the density of a symmetric one-dimensional log-concave r.v. X is bounded by $C/\|X\|_2$ (which follows by Theorem 2.2). In combination with (2) this ends the proof of the lemma. \square

It remains to prove Lemma 3.5. The argument follows the ideas introduced by Tao and Vu in [41] and relies on a bound on a distance between a single row of the matrix and the subspace spanned by some other k rows. Since contrary to the situations considered in [41, 1], we do not have independence between rows, a prominent role in the proof will be played by log-concavity and unconditionality, which will allow us to replace independence with upper bounds on the densities given in Theorem 2.2.

Lemma 4.1 (Distance to a random subspace). *Let A_n be an $n \times n$ random matrix with log-concave isotropic unconditional distribution and M_n a deterministic $n \times n$ matrix with $\|M_n\| \leq R\sqrt{n}$. Denote the rows of $A_n + M_n$ by Z_1, \dots, Z_n and let H be the space spanned by Z_1, \dots, Z_k ($k < n$). Then with probability at least $1 - 2n \exp(-c_R(n - k)^{1/3})$,*

$$\text{dist}(Z_{k+1}, H) \geq c_R \sqrt{n - k}.$$

Before we prove the above lemma let us show how it implies Lemma 3.5. The argument is due to Tao and Vu (see the proof of [41, Lemma 6.7]). We present it here for completeness.

Proof of Lemma 3.5. Consider $i \geq n^\gamma$. Let $k = n - \lfloor i/2 \rfloor$ and let B'_n be the $k \times n$ matrix with rows Z_1, \dots, Z_k (we use the notation from Lemma 4.1). By Cauchy interlacing inequalities we have $s_{n-j}(A_n + M_n) \geq s_{n-j}(B'_n)$ for $j \geq \lfloor i/2 \rfloor$. Let H_j , $j = 1, \dots, k$ be the subspace of \mathbb{C}^n spanned by all the rows of B'_n except for the j -th one. By [41, Lemma A4],

$$\sum_{j=1}^k s_j(B'_n)^{-2} = \sum_{j=1}^k \text{dist}(Z_j, H_j)^{-2}.$$

By Lemma 4.1, for each $j \leq k$, $\text{dist}(Z_j, H_j) \geq c_R \sqrt{n - k + 1}$ with probability at least $1 - 2n \exp(-c_R i^{1/3})$ (note that we can use the lemma since all its assumptions are preserved under permutation of rows of the matrix A_n). Thus by the union bound, with probability at least $1 - 2n^2 \exp(-c_R n^{\gamma/3})$, we get

$$\sum_{j=1}^k s_j(B'_n)^{-2} \leq C_R \frac{k}{i}.$$

On the other hand, the left-hand side above is at least $s_{n-i}(B'_n)^{-2}(k - n + i) \geq s_{n-i}(B'_n)^{-2}i/2$. This gives that with probability at least $1 - 2n^2 \exp(-c_R n^{\gamma/3})$,

$$s_{n-i}(A_n + M_n)^2 \geq s_{n-i}(B'_n)^2 \geq c_R \frac{i^2}{n - \lfloor n^\gamma/2 \rfloor},$$

which implies that $s_{n-i}(\frac{1}{\sqrt{n}}(A_n + M_n)) \geq c_R \frac{i}{n}$. We may now conclude the proof by taking the union bound over all $i \geq n^\gamma$ and adjusting the constants. \square

Proof of Lemma 4.1. Denote the rows of M_n by Y_1, \dots, Y_n . Recall that thanks to unconditionality we can assume that $A_n = [\varepsilon_{ij} X_{ij}]_{i,j \leq n}$, where $[X_{ij}]_{i,j \leq n}$ is log-concave, isotropic and unconditional. For simplicity in what follows we will write ε_j instead of $\varepsilon_{k+1,j}$.

With probability one $\dim(H) = k$. Let us however replace H by $K = \text{span}(H, Y_{k+1})$ and assume without loss of generality that this space is of dimension $k + 1$ (if not one can always choose a vector \tilde{Y} , measurable with respect to $\sigma(Z_1, \dots, Z_k, Y_{k+1})$ such that $K = \text{span}(H, \tilde{Y})$ is of dimension $k + 1$). Let e_1, \dots, e_{n-k-1} be an orthonormal basis in K^\perp and P be the orthogonal projection onto K^\perp . By e_{ij} we will denote the j -th coordinate of e_i .

Without loss of generality we can also assume that $k > n/2$ (otherwise we may change the constant c_R to $c_R/2$).

The proof will consist of several steps. First we will take advantage of independence between ε_{ij} 's and X_{ij} 's and use concentration of measure on the discrete cube to provide a lower bound on the distance which will be expressed in terms of X_{ij} 's and e_{ij} 's and will hold with high probability with respect to the Rademacher variables (i.e. conditionally on X_{ij} 's). Next we will use log-concavity to show that the random lower bound is itself with high probability bounded from below by $c_R \sqrt{n-k}$.

Step 1. Estimates with respect to Rademachers. We have

$$\begin{aligned} \text{dist}(X_{k+1}, K) &= |PX_{k+1}| = \left(\sum_{i=1}^{n-k-1} |\langle X_{k+1}, e_i \rangle|^2 \right)^{1/2} \\ &= \left(\sum_{i=1}^{n-k-1} \left| \sum_{j=1}^n X_{k+1,j} \varepsilon_j \bar{e}_{ij} \right|^2 \right)^{1/2} =: f(\varepsilon_1, \dots, \varepsilon_n). \end{aligned} \quad (3)$$

The function f is a semi-norm, L -Lipschitz with

$$L = \sup_{x \in S^{n-1}} |f(x)| = \sup_{x \in S^{n-1}} |P(X_{k+1,i} x_i)_{i=1}^n| \leq \sup_{x \in S^{n-1}} \sqrt{\sum_{i=1}^n X_{k+1,i}^2 x_i^2} \leq \max_{i \leq n} |X_{k+1,i}|.$$

Moreover, using the Khintchin-Kahane inequality we get

$$\mathbb{E}_\varepsilon f(\varepsilon_1, \dots, \varepsilon_n) \geq c(\mathbb{E}_\varepsilon f(\varepsilon_1, \dots, \varepsilon_n)^2)^{1/2} \geq c \left(\sum_{j=1}^n X_{k+1,j}^2 \left(\sum_{i=1}^{n-k-1} |e_{ij}|^2 \right) \right)^{1/2}.$$

Since $\text{dist}(Z, H) \geq \text{dist}(X_{k+1}, K)$, by Talagrand's concentration inequality on the discrete cube (see [39]) we get for some absolute constant $c > 0$,

$$\begin{aligned} \mathbb{P}_\varepsilon \left(\text{dist}(Z_{k+1}, H) \leq c \left(\sum_{j=1}^n X_{k+1,j}^2 \left(\sum_{i=1}^{n-k-1} |e_{ij}|^2 \right) \right)^{1/2} \right) \\ \leq 2 \exp \left(-c \frac{\sum_{j=1}^n X_{k+1,j}^2 \left(\sum_{i=1}^{n-k-1} |e_{ij}|^2 \right)}{\max_{i \leq n} X_{k+1,i}^2} \right). \end{aligned} \quad (4)$$

Step 2. Lower bounds on coordinates on e_i . Let $\text{Sparse}(\delta)$ denote the set of δn sparse vectors in \mathbb{C}^n , i.e. vectors with at most δn nonzero coordinates. Let $S_{\mathbb{C}}^{n-1}$ be the unit Euclidean ball in \mathbb{C}^n and define the set of compressible and incompressible vectors by

$$\text{Comp}(\delta, \rho) = \{x \in S_{\mathbb{C}}^{n-1} : \text{dist}(x, \text{Sparse}(\delta)) \leq \rho\} \quad \text{and} \quad \text{Incomp}(\delta, \varepsilon) = S_{\mathbb{C}}^{n-1} \setminus \text{Comp}(\delta, \varepsilon).$$

We will now show that with high probability for each $i \leq n-k-1$, $e_i \in \text{Incomp}(\delta, \varepsilon)$ (with δ, ε depending only on R).

We will follow the by now standard approach (see [37, 31]) and consider first the set of sparse vectors. Let A', M', B' be $k \times n$ matrices with rows resp. $(X_i)_{i \leq k}, (Y_i)_{i \leq k}, (Z_i)_{i \leq k}$ and denote by X'_i, Y'_i, Z'_i ($i = 1, \dots, n$) their columns. Note that for any real vector $x \in S^{n-1}$, the random vector

$$S = \sum_{i=1}^n x_i X'_i$$

is a log concave isotropic unconditional random vector in \mathbb{R}^k (it is log-concave as a linear image of a log-concave vector, unconditionality and isotropicity can be directly verified).

Therefore, by Lemma 2.2, it has a density bounded by C^k . Thus for any deterministic vector $v \in \mathbb{R}^k$,

$$\mathbb{P}(S \in v + 2r\sqrt{k}B_2^k) \leq C^k 2^k r^k k^{k/2} \text{vol}(B_2^k) \leq C^k r^k, \quad (5)$$

where B_2^k is the unit Euclidean ball in \mathbb{R}^k . Consider now any $x \in S_{\mathbb{C}}^{n-1}$ and let $x' = \Re x$, $x'' = \Im x$. We have

$$B'x = \sum_{i=1}^n x'_i X'_i + \Re\left(\sum_{i=1}^n x_i Y'_i\right) + i \sum_{i=1}^n x''_i X'_i + i \Im\left(\sum_{i=1}^n x_i Y'_i\right).$$

Setting $v' = -\Re \sum_{i=1}^n x_i Y'_i$ and $v'' = -\Im \sum_{i=1}^n x_i Y'_i$ we get

$$\mathbb{P}(|B'x| \leq 2r\sqrt{k}) \leq \min\left(\mathbb{P}\left(\left|\sum_{i=1}^n x'_i X'_i - v'\right| \leq 2r\sqrt{k}\right), \mathbb{P}\left(\left|\sum_{i=1}^n x''_i X'_i - v''\right| \leq 2r\sqrt{k}\right)\right).$$

At least one of the vectors x' , x'' has Euclidean norm not smaller than $2^{-1/2}$. Thus using (5), we get

$$\mathbb{P}(|B'x| \leq 2r\sqrt{k}) \leq C^k r^k. \quad (6)$$

Note that the set $\text{Sparse}(\delta) \cap S_{\mathbb{C}}^{n-1}$ admits an ε -net \mathcal{N} of cardinality

$$\binom{n}{\lfloor \delta n \rfloor} (3/\varepsilon)^{2\lfloor \delta n \rfloor} \leq \left(\frac{C}{\varepsilon^2 \delta}\right)^{\delta n}.$$

(This can be easily seen by using volumetric estimates for each choice of the support). By (6) and the union bound with probability at least

$$1 - r^k C^n \left(\frac{1}{\varepsilon^2 \delta}\right)^{\delta n},$$

for all $x \in \mathcal{N}$,

$$|B'x| > 2r\sqrt{k}.$$

If $\varepsilon \leq 1/2$ then on the above event we have,

$$|B'x| > r\sqrt{k} - 2\varepsilon \|B'\|$$

for all $x \in \text{Comp}(\delta, \varepsilon)$. Indeed if $x \in \text{Comp}(\delta, \varepsilon)$, then there exists $y \in \text{Sparse}(\delta)$ such that $|x - y| \leq \varepsilon$. But $|y| \geq 1 - \varepsilon$ then and therefore $|B'y| \geq (1 - \varepsilon)|B' \frac{y}{|y|}| \geq (1 - \varepsilon)(2r\sqrt{k} - \varepsilon \|B'\|)$, since $y/|y| \in \text{Sparse}(\delta) \cap S_{\mathbb{C}}^{n-1}$ (and so it can be ε -approximated by a vector from \mathcal{N}). Now $|B'x| \geq |B'y| - \varepsilon \|B'\| \geq r\sqrt{k} - 2\varepsilon \|B'\|$.

Thus by Lemma 3.2 and the assumption $k > n/2$ we have

$$|B'x| > (r/2 - 3(C + R)\varepsilon)\sqrt{n}$$

for all $x \in \text{Comp}(\delta, \varepsilon)$, with probability at least

$$1 - r^{n/2} C^n \left(\frac{1}{\varepsilon^2 \delta}\right)^{\delta n} - 2 \exp(-c(R + C)\sqrt{n}).$$

If we set $\varepsilon = r/(12(C + R))$, we get for $r \in (0, 1)$ and δ small enough (depending only on R),

$$|B'x| > r\sqrt{n}/4 > 0$$

for all $x \in \text{Comp}(\delta, \varepsilon)$, with probability at least

$$\begin{aligned} & 1 - r^{n/2} C^n \left(\frac{144(R + C)^2}{r\delta}\right)^{\delta n} - 2 \exp(-c(R + C)\sqrt{n}) \\ & \geq 1 - C^n r^{n/4} - 2 \exp(-c(R + C)\sqrt{n}). \end{aligned}$$

Now for r sufficiently small, the right hand side above is greater than $1 - 2\exp(-c\sqrt{n})$. In particular we have shown that there exist $\delta, \varepsilon > 0$ depending only on R such that with probability at least $1 - 2\exp(-c\sqrt{n})$,

$$\inf_{x \in \text{Comp}(\delta, \varepsilon)} |B'x| > 0$$

and in consequence (since $B'e_i = 0$)

$$e_i \in \text{Incomp}(\delta, \varepsilon) \quad (7)$$

for $i = 1, \dots, n - k - 1$.

Step 3. Estimating conditional expectation. From [37, Lemma 3.4] we get that whenever $x \in \text{Incomp}(\delta, \varepsilon)$, then there exists a set $I \subseteq \mathbb{C}^n$ of cardinality at least $\frac{1}{2}\varepsilon^2\delta n$, such that for all $i \in I$, $|x_i| \geq \frac{\varepsilon}{\sqrt{2n}}$ (the lemma is proved in [37] in the real case, but the proof works as well for \mathbb{C}^n , alternatively one may formally pass to the complex case by identifying \mathbb{C}^n with \mathbb{R}^{2n} , which will just slightly change the constants).

Using this fact and (7), we get that with probability at least $1 - 2\exp(-c\sqrt{n})$, for $i = 1, \dots, n - k - 1$, there exists a set $I_i \subseteq \{1, \dots, n\}$ of cardinality at least αn (where $\alpha > 0$ depends only on R) such that $|e_{ij}|^2 \geq \varepsilon^2/2n$ for all $j \in I_i$. We will now prove that for some constant $\rho > 0$, depending only on R , with probability at least $1 - 2\exp(-\alpha n)$, the set $J = \{j : |X_{k+1,j}| \geq \rho\}$ satisfies

$$|J| > (1 - \alpha/2)n. \quad (8)$$

Thus we will obtain that for some β , depending only on R , with probability $1 - 2\exp(-c_R\sqrt{n})$,

$$\sum_{j=1}^n X_{k+1,j}^2 \sum_{i=1}^{n-k-1} |e_{ij}|^2 \geq \sum_{i=1}^{n-k-1} \sum_{j \in I_i \cap J} X_{k+1,j}^2 |e_{ij}|^2 \geq (n - k - 1) \frac{\alpha n}{2} \rho^2 \frac{\varepsilon^2}{2n} \geq \beta(n - k - 1). \quad (9)$$

To prove (8) we note that for every set $I \subseteq \{1, \dots, n\}$ of cardinality m , the random vector $(X_{k+1,i})_{i \in I}$ is isotropic, log-concave and unconditional and hence by Theorem 2.2 has a density bounded by C^m . Thus

$$\mathbb{P}(|X_{k+1,i}| \leq \rho \text{ for all } i \in I) \leq 2^m C^m \rho^m.$$

Taking the union bound over all sets of cardinality $m = \lfloor \alpha n/2 \rfloor$ we obtain

$$\mathbb{P}(|\{j : |X_{k+1,j}| \geq \rho\}| \leq (1 - \alpha/2)n) \leq \binom{n}{m} 2^m C^m \rho^m \leq 2^{\alpha n} C^{\alpha n} e^{C\alpha n \log(2/\alpha)} \rho^{\lfloor \alpha n/2 \rfloor},$$

which gives (8) for $\rho = \exp(-C \log(2/\alpha))$.

Step 4. Conclusion of the proof. Without loss of generality we may assume that $n - k > C$ (otherwise we can make the bound on probability in the statement of the lemma trivial, by playing with the constant c_R). Note that by Theorem 2.1 and the fact that $X_{k+1,i}$ is log-concave of mean zero and variance one,

$$\mathbb{P}(\max_{i \leq n} X_{k+1,i}^2 > (n - k)^{2/3}) \leq 2n \exp(-c(n - k)^{1/3}).$$

Combining this estimate with (9) we get

$$\mathbb{P}\left(\sum_{j=1}^n X_{k+1,j}^2 \sum_{i=1}^{n-k-1} |e_{ij}|^2 \geq \beta(n - k)/2, \max_{i \leq n} X_{k+1,i}^2 \leq (n - k)^{2/3}\right) \geq 1 - 4n \exp(-c_R(n - k)^{1/3}).$$

Denote the event above by U . Using the above estimate together with (4) and the Fubini theorem, we get

$$\begin{aligned} \mathbb{P}(\text{dist}(Z_{k+1}, H) \leq c\sqrt{\beta/2}\sqrt{n-k}) &\leq \mathbb{E}\left(\mathbf{1}_U \mathbb{P}_\varepsilon(\text{dist}(Z_{k+1}, H) \leq c\sqrt{\beta/2}\sqrt{n-k})\right) + \mathbb{P}(U^c) \\ &\leq 6n \exp(-c_R(n-k)^{1/3}). \end{aligned}$$

To prove the lemma it is now enough to adjust the constants. \square

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